

Computational complexity of solving polynomial differential equations over unbounded domains with non-rational coefficients

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Abstract

In this note, we extend the result of [PG16] about the complexity of solving polynomial differential equations over unbounded domains to work with non-rational input. In order to deal with arbitrary input, we phrase the result in framework of Computable Analysis [Ko91]. As a side result, we also get a uniform result about complexity of the operator, and not just about the solution.

The complexity of solving this kind of differential equation has been heavily studied over compact domains but there are few results over unbounded domains. In [PG16] we studied the complexity of this problem over unbounded domains and obtained a bound that involved the length of the solution curve. Unfortunately, the result was written for rational inputs only. In this note, we extend it to work with any numbers, in the framework of Computable Analysis. To do so, we will need to recall a few lemmas and introduce some notation. For any continuous function y , define

$$\text{Int}_y(a, b, \varepsilon) = \int_a^b k \Sigma p \max(1, \varepsilon + \|y(u)\|)^{k-1} du$$

and

$$\ell_y(a, b) = \int_a^b \Sigma p \max(1, \|y(u)\|)^k du.$$

For any multivariate polynomial $p(x) = \sum_{|\alpha| \leq k} a_\alpha x^\alpha$, we call k the degree and denote the sum of the norm of the coefficients by $\Sigma p = \sum_{|\alpha| \leq k} \|a_\alpha\|$. Note that a vector of polynomials can be identified to a vector with vector coefficients (i.e. $\mathbb{K}^d[\mathbb{R}^n]$ is isomorphic to $(\mathbb{K}[\mathbb{R}^n])^d$) and always make this transformation implicitly below. For such a polynomial p and $\eta \geq 0$, we call a η -relative-approximation of p any polynomial $\tilde{p} = \sum_{|\alpha| \leq k} \tilde{a}_\alpha x^\alpha$ with the same degree such that $\|\tilde{a}_\alpha - a_\alpha\| \leq \eta \|a_\alpha\|$ for all $|\alpha| \leq k$. It follows almost by definition that:

Lemma 1. *If \tilde{p} is a η -relative-approximation of $p \in \mathbb{R}^n[\mathbb{R}^d]$ then for all $x \in \mathbb{R}^d$ we have $\|\tilde{p}(x) - p(x)\| \leq \eta \Sigma p \max(1, \|x\|)^k$ where k is the degree of p .*

We also recall the following simple lemma about polynomials.

Lemma 2 ([PG16]). *Let $p \in \mathbb{R}^n[\mathbb{R}^d]$ and k its degree. For all $a, b \in \mathbb{R}^d$ we have*

$$\|p(b) - p(a)\| \leq k \Sigma p \|b - a\| \max(\|a\|, \|b\|)^{k-1}.$$

We will need to quantify the divergence between two PIVPs with slightly different initial conditions and errors in the coefficients of the polynomials.

Proposition 3. Let $I = [a, b]$ be an interval, $p \in \mathbb{R}^n[\mathbb{R}^n]$ and k its degree, $y_0, \tilde{y}_0 \in \mathbb{R}^n$ and \tilde{p} a η -relative-approximation of p for some $\eta \geq 0$. Assume that $y, \tilde{y} : I \rightarrow \mathbb{R}^n$ satisfies for all $t \in I$

$$\begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \end{cases} \quad \begin{cases} \tilde{y}(0) = \tilde{y}_0 \\ \tilde{y}'(t) = \tilde{p}(\tilde{y}(t)) \end{cases}.$$

For any $\varepsilon > 0$ and $t \in I$, let

$$\mu_\varepsilon(t) = (\|\tilde{y}_0 - y_0\| + \eta \ell_y(a, t)) \exp((1 + \eta) \text{Int}_y(a, t, \varepsilon)).$$

If $\mu_\varepsilon(t) < \varepsilon$ then $\|z(t) - y(t)\| \leq \mu_\varepsilon(t)$. Furthermore, if the existence of \tilde{y} is not known, then $\mu_\varepsilon(t) < \varepsilon$ implies that \tilde{y} exists over $[a, b]$.

Proof. Let $\psi(t) = \|\tilde{y}(t) - y(t)\|$. For any $t \in I$, we have

$$\psi(t) \leq \psi(a) + \int_a^t \|\tilde{p}(\tilde{y}(u)) - p(y(u))\| du.$$

Note that $\Sigma \tilde{p} \leq (1 + \eta) \Sigma p$ and apply Lemmas 1 and 2 to get, for $N(u) = \|y(u)\| + \psi(u)$, that

$$\|\tilde{p}(\tilde{y}(u)) - p(y(u))\| \leq \eta \Sigma p \max(1, \|y(u)\|)^k + k(1 + \eta) \Sigma p N^{k-1}(u) \psi(u).$$

Putting everything together, we have

$$\psi(t) \leq \psi(a) + \int_a^t \eta \Sigma p \max(1, \|y(u)\|)^k du + \int_a^t (1 + \eta) k \Sigma p N^{k-1}(u) \psi(u) du.$$

Apply the Generalized Gronwall's Inequality, using that the integral of non-negative values is non-decreasing, to get

$$\psi(t) \leq \left(\|\tilde{y}_0 - y_0\| + \int_a^t \eta \Sigma p \max(1, \|y(u)\|)^k du \right) \exp \left(\int_a^t (1 + \eta) k \Sigma p N^{k-1}(u) du \right).$$

Define $t_1 = \max \{t \in I \mid \forall u \in [a, t], \psi(u) \leq \varepsilon\}$ which is well-defined as the maximum of a closed and non-empty set (a belongs to it). Then for all $t \in [0, t_1]$, $N(t) \leq \|y(t)\| + \varepsilon$ and thus:

$$\begin{aligned} \psi(t) &\leq \left(\|\tilde{y}_0 - y_0\| + \int_a^t \eta \Sigma p \max(1, \|y(u)\|)^{k-1} du \right) \exp \left(\int_a^t (1 + \eta) k \Sigma p (\|y(u)\| + \varepsilon)^{k-1} du \right) \\ &\leq (\|\tilde{y}_0 - y_0\| + \eta \ell_y(0, t)) \exp((1 + \eta) \text{Int}_y(0, t, \varepsilon)) \\ &\leq \mu_\varepsilon(t). \end{aligned}$$

We will show by contradiction that $t_1 = b$, which proves the result. Assume by contradiction that $t_1 < b$. Then by continuity of ψ and because $\psi(a) = \mu(a) < \varepsilon$, there exists $t_0 \leq t_1$ such that $\psi(t_0) = \varepsilon$. But then $t_0 \in [0, t_1]$ so $\psi(t_0) \leq \mu(t_0) < \varepsilon$ by hypothesis, which is impossible.

To show the existence, assume by contradiction \tilde{y} does not exist over $[a, b]$. Apply Cauchy-Lipschitz theorem to get a maximal solution \tilde{y} that exists over $[a, c[$ but not $[a, c]$ where $c \in [a, b]$. It is a well-known fact that $\|\tilde{y}(t)\| \rightarrow +\infty$ as $t \rightarrow c$. Since $[a, b]$ is compact, y is bounded over $[a, b]$. It follows that $\|\tilde{y}(t) - y(t)\| \rightarrow +\infty$ as $t \rightarrow c$. Thus by continuity, there exists $d \in [a, c[$ such that $\|\tilde{y}(d) - y(d)\| = \varepsilon$. But then \tilde{y} exists over $[a, d]$ so we can apply the above reasoning over $[a, d]$ to get that $\|\tilde{y}(d) - y(d)\| \leq \mu_\varepsilon(d)$ since $\mu_\varepsilon(d) \leq \mu_\varepsilon(b) < \varepsilon$. It follows that $\|\tilde{y}(d) - y(d)\| < \varepsilon$ which is impossible. \square

We will need a result on the growth of the PIVP that only involves the initial condition.

Proposition 4. Let $I = [a, b]$ be an interval, $p \in \mathbb{R}^n[\mathbb{R}^n]$ and k its degree and $y_0 \in \mathbb{R}^n$. Assume that $y : I \rightarrow \mathbb{R}^n$ satisfies for all $t \in I$ that

$$y(a) = y_0 \quad y'(t) = p(y(t)),$$

then

$$\|y(t) - y(a)\| \leq \frac{\alpha M |t - a|}{1 - M |t - a|}$$

for every t such that $M |t - a| < 1$ where $M = (k - 1) \Sigma p \alpha^{k-1}$ and $\alpha = \max(1, \|y_0\|)$.

Proof. This is a consequence of Theorem 5 (Taylor approximation for PIVP) in [PG16], restating an original result in [WWS⁺06]. \square

We now recall the complexity result in [PG16]. For reasons that will appear later, we will use the algorithm with “hint” rather than the full algorithm.

Theorem 5 (Solving PIVPs with hint, [PG16]). *There exists an algorithm \mathcal{A} such that the following holds. Let $a, b \in \mathbb{Q}$, $p \in \mathbb{Q}^n[\mathbb{R}^n]$ and k its degree and $y_0 \in \mathbb{Q}^n$. Assume that $y : [a, b] \rightarrow \mathbb{R}^n$ satisfies for all $t \in [a, b]$ that*

$$y(a) = y_0 \quad y'(t) = p(y(t)).$$

Let $I, \varepsilon \in \mathbb{Q}$ and $x = \mathcal{A}(a, y_0, p, b, \varepsilon, I)$, then

- either $x = \perp$ or $\|y(b) - x\| \leq \varepsilon$,
- if $I \geq 6 \text{Int}_y(a, b, \varepsilon)$ then $x \neq \perp$,
- if $I < \text{Int}_y(a, b, \varepsilon)$ then $x = \perp$,
- the algorithm computes x in time bounded in by

$$\text{poly}(k, I, \log \ell_y(a, b), \log \|y_0\|, \log \Sigma p, -\log \varepsilon)^n.$$

Proof. This is a consequence of various results in [PG16]. The first two points follows from Lemma 10 (Algorithm is correct) and the third one follows from the proof of Lemma 10 (but is not stated in the Lemma itself). The fourth point is a consequence of Lemma 14 (Complexity of SolvePIVPVariable). \square

For technical reasons, the previous lemma is not entirely satisfactory because the hint I is related to Int_y but we would prefer that it relates to ℓ_y . This is possible thanks to a small trick.

Lemma 6. *There exists an algorithm \mathcal{B} such that the following holds. Let $a, b \in \mathbb{Q}$, $p \in \mathbb{Q}^n[\mathbb{R}^n]$ and k its degree and $y_0 \in \mathbb{Q}^n$. Assume that $y : [a, b] \rightarrow \mathbb{R}^n$ satisfies for all $t \in [a, b]$ that*

$$y(0) = y_0 \quad y'(t) = p(y(t)).$$

Let $L, \varepsilon \in \mathbb{Q}$ and $x = \mathcal{B}(a, y_0, p, b, \varepsilon, L)$, then

- either $x = \perp$ or $\|y(b) - x\| \leq \varepsilon$,
- if $L \geq 12(k + 1)\ell_y(a, b)$ then $x \neq \perp$,
- if $L < \ell_y(a, b)$ then $x = \perp$,
- the algorithm computes x in time bounded in by

$$\text{poly}(k, L, \log \ell_y(a, b), \log \|y_0\|, \log \Sigma p, -\log \varepsilon)^n.$$

Furthermore, even if there no solution y to the system over $[a, b]$, the algorithm always returns \perp in time bounded by

$$\text{poly}(k, L, \log \|y_0\|, \log \Sigma p, -\log \varepsilon)^n.$$

Proof. Let \mathcal{A} be the algorithm from Theorem 5. The hint of \mathcal{A} is related to Int_y which contains the integral of $\max(1, \|y(t)\|)^{k-1}$. On the other hand, we would like to related to ℓ_y which contains the integral of $\max(1, \|y(t)\|)^k$. So if we could increase the degree artificially by one, without changing the complexity too much, we would almost have what we want. The idea is to add one component that will always be 0 but with a polynomial of degree $k+1$. One possibility is $z' = z^{k+1}$ with $z(0) = 0$ but it will be more convenient to take $z' = \Sigma p z^{k+1}$.

Without loss of generality, we assume that $\varepsilon \leq \frac{1}{4k}$. Given the hypothesis of the lemma, let

$$z_0 = (y_0, 0), \quad q(y, z) = (p(y), \Sigma p z^{k+1}).$$

and define

$$\mathcal{B}(a, y_0, p, b, \varepsilon, L) = \mathcal{A}(a, z_0, q, b, \varepsilon, L)_{1..n}.$$

It is clear from the definition that the only solution of

$$z(0) = z_0 \quad z'(t) = q(z(t))$$

is of the form $z(t) = (y(t), 0)$. We will now check that \mathcal{B} satisfies the claim. Let $x = \mathcal{A}(a, y_0, p, b, \varepsilon, L)$. First, recall that Σq is the maximum of all components of q , and since $\Sigma(z \mapsto \Sigma p z^{k+1}) = \Sigma p$ we get that $\Sigma q = \Sigma p$. Furthermore, q is of degree $k+1$ and $\|z(t)\| = \|y(t)\|$ for all $t \in [a, b]$.

- By definition of \mathcal{A} , either $x = \perp$ (and thus $x_{1..n} = \perp$) or $\|x - z(t)\| \leq \varepsilon$, but since $z(t) = (y(t), 0)$ then $\|x_{1..n} - y(t)\| \leq \varepsilon$.
- If $L \geq 12(k+1)\ell_y(a, b)$ then

$$\begin{aligned} 6 \text{Int}_z(a, b, \varepsilon) &= 6 \int_a^b (k+1) \Sigma q \max(1, \varepsilon + \|z(u)\|)^{(k+1)-1} du \\ &= 6(k+1) \int_a^b \Sigma p \max(1, \varepsilon + \|y(u)\|)^k du \\ &\leq 6(k+1)(1+\varepsilon)^k \int_a^b \Sigma p \max(1, \|y(u)\|)^k du \\ &\leq 6(k+1)(1+\frac{1}{4k})^k \ell_y(a, b) \\ &\leq 12(k+1)\ell_y(a, b) \\ &\leq L. \end{aligned}$$

Thus $x \neq \perp$ by Theorem 5.

- If $L < \ell_y(a, b)$ then

$$\begin{aligned} L &< \int_a^b \Sigma p \max(1, \|y(u)\|)^k du \\ &= \int_a^b \Sigma q \max(1, \|z(u)\|)^k du \\ &\leq \int_a^b (k+1) \Sigma q \max(1, \varepsilon + \|z(u)\|)^{(k+1)-1} du \\ &= \text{Int}_z(0, t, \varepsilon). \end{aligned}$$

Thus $x = \perp$ by Theorem 5.

- By Theorem 5, the complexity is bounded by

$$\text{poly} \left(k+1, L, \log \ell_z(a, b), \log \|z_0\|, \log \Sigma q, -\log \varepsilon \right)^{n+1}.$$

Recall that for any $t \in [a, b]$ we have

$$\begin{aligned} \|y(t)\| &\leq \|y_0\| + \int_0^t \|y'(u)\| du \\ &= \|y_0\| + \int_0^t \|p(y(u))\| du \\ &\leq \|y_0\| + \int_0^t \Sigma p \max(1, \|y(u)\|)^k du \\ &= \|y_0\| + \ell_y(0, t) \\ &\leq \|y_0\| + \ell_y(0, b). \end{aligned}$$

Thus

$$\begin{aligned} \ell_z(a, b) &= \int_a^b \Sigma q \max(1, \|z(u)\|)^{k+1} du \\ &= \int_a^b \Sigma p \max(1, \|y(u)\|)^{k+1} du \\ &\leq \max(1, \|y_0\| + \ell_y(a, b)) \int_a^b \Sigma p \max(1, \|y(u)\|)^k du \\ &\leq \max(1, \|y_0\| + \ell_y(a, b)) \ell_y(a, b). \end{aligned}$$

It follows that the complexity is bounded by

$$\text{poly} \left(k, L, \log \ell_y(a, b), \log \|y_0\|, \log \Sigma p, -\log \varepsilon \right)^n.$$

The extra statement is a consequence of two facts. First, disregarding the existence or not of y , if $b' < b$ and $\mathcal{A}(a, z_0, q, b', \varepsilon, L) = \perp$ then $\mathcal{A}(a, z_0, q, b, \varepsilon, L) = \perp$. This is a consequence of the fact that the algorithm does not use b in any intermediate computation except to check if it has reached time b . In other words, the algorithm will perform exactly the same on the two instances and thus return \perp in both. We refer the reader to Algorithm 11 in [PG16] to check the details of this claim. Furthermore, it follows from this that the running of the algorithm on both instances is the same (they execute exactly the same number of instructions).

Second, by the Cauchy-Lipschitz theorem, there exists a maximal solution y whose domain is open and contains a neighbourhood of a . Thus there exists a $c \in]a, b]$ such that y is defined over $[a, c[$ but not in c . It is a well-known fact that $\|y(t)\| \rightarrow +\infty$ as $t \rightarrow c$. Since, as we saw above, $\|y(t)\| \leq \|y_0\| + \ell_y(0, t)$, it follows that $\ell_y(a, t) \rightarrow +\infty$ as $t \rightarrow c$. Thus by continuity, there exists $b' \in [a, c[$ such that $\ell_y(a, b') = L + 1$. But then, by the third point above (and since y exists over $[a, b']$),

$$\mathcal{A}(a, z_0, q, b', \varepsilon, L) = \perp.$$

And since $b' < b$, it follows that $\mathcal{A}(a, z_0, q, b, \varepsilon, L) = \perp$ by the claim above. Furthermore, since we saw earlier that the complexity of both instances is the same, it follows that it returns \perp in time bounded by

$$\text{poly} \left(k, L, \log \ell_y(a, b'), \log \|y_0\|, \log \Sigma p, -\log \varepsilon \right)^n.$$

which satisfies the claim since $\ell_y(a, b') = L + 1$. \square

We are now ready to state and prove a result about the complexity of solving PIVPs for any inputs.

Theorem 7 (Complexity of Solving PIVPs). *Let $I = [a, b]$ be an interval, $p \in \mathbb{R}^d[\mathbb{R}^d]$ and k its degree and $y_0 \in \mathbb{R}^d$. Assume that $y : I \rightarrow \mathbb{R}^d$ satisfies for all $t \in I$ that*

$$y(a) = y_0 \quad y'(t) = p(y(t)), \quad (1)$$

then $y(b)$ can be computed with precision $2^{-\mu}$ in time bounded by

$$\text{poly}(k, \ell_y(a, b), \log \|y_0\|, \log \Sigma p, \mu)^d. \quad (2)$$

More precisely, there exists a Turing machine \mathcal{M} such that for any oracle \mathcal{O} representing¹ (a, y_0, p, b) and any $\mu \in \mathbb{N}$, $\|\mathcal{M}^{\mathcal{O}}(\mu) - y(b)\| \leq 2^{-\mu}$ where y satisfies (1), and the number of steps of the machine is bounded by (2) for all such oracles.

Proof. Let \mathcal{B} be the algorithm from Lemma 6. Without loss of generality we assume that $a \in \mathbb{Q}$ (since we can always replace a by 0 and b by $b - a$). Let \mathcal{O} be an oracle for a, y_0, p and b (where p is represented by the finite list of its coefficients) and μ the input of the machine. Let $\varepsilon \in \mathbb{Q}$ such that $\varepsilon < e^{-\mu - \ln 3}$. Define, for all $n \in \mathbb{N}$:

- $L_n = n$,
- $\nu_n = e^{-4kL_n - \ln 2} \varepsilon$,
- $y_0^{(n)} \in \mathbb{Q}^n$ be such that $\|y_0^{(n)} - y_0\| \leq \nu_n$,
- $\eta_n \in \mathbb{Q}^n$ be such that $\eta_n \leq \frac{\nu_n}{L_n}$ and $\eta_n < 1$,
- $p^{(n)}$ be a η_n -relative-approximation of p ,
- $t^{(n)} \in \mathbb{Q}$ be such that $t^{(n)} \leq b$ and

$$b - t^{(n)} \leq \frac{\varepsilon}{2k\Sigma p \max(1, \|y_0\| + L_n)^k}.$$

Finally define the sequence

$$x_n = \mathcal{B}(a, y_0^{(n)}, p^{(n)}, t^{(n)}, \varepsilon, L_n)$$

and let $y^{(n)}$ be the maximal solution of

$$y^{(n)}(a) = y_0^{(n)} \quad y^{(n)'} = p^{(n)}(y^{(n)}).$$

Note that by the Cauchy-Lipschitz theorem, we know such a solution exists but it may not exist over $[a, b]$. Note, and this is a consequence of Lemma 6, that we can safely apply \mathcal{B} to a system even if we don't know that its solution exists over $[a, b]$.

First, we claim that if $L_n \geq \ell_y(a, b)$ then $y^{(n)}$ exists over $[a, t^{(n)}]$ and $\|y(u) - y^{(n)}(u)\| \leq \varepsilon$ for all $u \in [a, t^{(n)}]$. Indeed, assume that $L_n \geq \ell_y(a, b)$. Then

$$L_n \geq \ell_y(a, b) \geq \ell_y(a, t^{(n)}).$$

Let

$$\mu_\varepsilon(t) = \left(\|y_0^{(n)} - y_0\| + \eta_n \ell_y(a, t) \right) \exp((1 + \eta_n) \text{Int}_y(a, t, \varepsilon)).$$

Apply Lemma 13 (Relationship between Int and Len) in [PG16] to get that

$$\text{Int}_y(a, t, \varepsilon) \leq 2k\ell_y(a, t).$$

¹See [Ko91] for more details. In short, the machine can ask arbitrary approximation of a, y_0, p and b to the oracle. The polynomial is represented by the finite list of coefficients.

It follows that

$$\begin{aligned}
\mu_\varepsilon(t^{(n)}) &\leq \left(\|y_0^{(n)} - y_0\| + \eta_n \ell_y(a, t^{(n)}) \right) \exp \left((1 + \eta_n) 2k \ell_y(a, t^{(n)}) \right) \\
&\leq (\nu_n + \eta_n L_n) \exp((1 + \eta_n) 2k L_n) \\
&\leq 2\nu_n \exp(4k L_n) \\
&< \varepsilon.
\end{aligned}$$

Apply Proposition 3 to get that $y^{(n)}$ exists over $[a, t^{(n)}]$. For all $u \in [a, t^{(n)}]$, note that $\mu_\varepsilon(u) \leq \mu_\varepsilon(t^{(n)}) < \varepsilon$ and apply Proposition 3 again over $[a, u]$ to get that

$$\|y(u) - y^{(n)}(u)\| \leq \varepsilon.$$

Second, we claim that if $x_n \neq \perp$ then $\|x_n - y(b)\| \leq e^{-\mu}$. Indeed, by Lemma 6, if $x_n \neq \perp$ then it must be the case that

$$L_n \geq \ell_y(a, b).$$

Apply the first claim to get that $y^{(n)}$ exists over $[a, t^{(n)}]$ and that

$$\|y(t^{(n)}) - y^{(n)}(t^{(n)})\| \leq \varepsilon.$$

Apply Lemma 6 to get that

$$\|x_n - y^{(n)}(t^{(n)})\| \leq \varepsilon.$$

It remains to see the relationship between $y(b)$ and $y(t^{(n)})$. Recall that

$$\|y(t^{(n)})\| \leq \|y_0\| + \ell_y(a, t^{(n)}) \leq \|y_0\| + L_n.$$

Let $M = (k-1)\Sigma p \alpha^{k-1}$ and $\alpha = \max(1, \|y(t^{(n)})\|)$. Note that $\alpha \leq \max(1, \|y_0\| + L_n)$. It follows by definition of $t^{(n)}$ that

$$\begin{aligned}
M|t - t^{(n)}| &= (k-1)\Sigma p \alpha^{k-1} |t - t^{(n)}| \\
&\leq k\Sigma p \max(1, \|y_0\| + L_n)^{k-1} |t - t^{(n)}| \\
&\leq \frac{\varepsilon}{2 \max(1, \|y_0\| + L_n)} \\
&\leq \frac{1}{2} < 1.
\end{aligned}$$

Thus we can apply Proposition 4 to y with $a = t^{(n)}$ to get that

$$\|y(b) - y(t^{(n)})\| \leq \frac{\alpha M |b - t^{(n)}|}{1 - M |b - t^{(n)}|}.$$

Consequently

$$\begin{aligned}
\|y(b) - y(t^{(n)})\| &\leq \frac{\alpha M |t - t^{(n)}|}{1 - M |t - t^{(n)}|} \\
&\leq \frac{\alpha \frac{\varepsilon}{2 \max(1, \|y_0\| + L_n)}}{1 - 1/2} \\
&\leq \varepsilon.
\end{aligned}$$

Putting everything together, we get that

$$\|x_n - y(b)\| \leq 3\varepsilon \leq e^{-\mu}.$$

Third, we claim that if $L_n \geq 48(k+1)\ell_y(a, b)$ then $x_n \neq \perp$. Indeed, assume that this is the case. Then in particular $L_n \geq \ell_y(a, b)$ so by the first fact, $y^{(n)}$ exists over $[a, t^{(n)}]$ and for all $t \in [a, t^{(n)}]$ we have

$$\|y(t) - y^{(n)}(t)\| \leq \varepsilon.$$

It follows from this that

$$\begin{aligned} \ell_{y^{(n)}}(a, t^{(n)}) &= \int_a^{t^{(n)}} \Sigma p^{(n)} \max\left(1, \|y^{(n)}(u)\|\right)^k du \\ &\leq \int_a^{t^{(n)}} (1 + \eta_n) \Sigma p \max(1, \|y(u)\| + \varepsilon)^k du \\ &\leq (1 + \eta_n)(1 + \varepsilon)^k \int_a^{t^{(n)}} \Sigma p \max(1, \|y(u)\|)^k du \\ &\leq 2(1 + \frac{1}{4k})^k \ell_y(a, t^{(n)}) \\ &\leq 4\ell_y(a, b). \end{aligned}$$

Thus

$$L_n \geq 48k\ell_y(a, b) \geq 12(k+1)\ell_{y^{(n)}}(a, t^{(n)})$$

and by Lemma 6, $x_n \neq \perp$.

Now consider the algorithm that computes the sequence $(x_n)_n$ and returns the first $x_n \neq \perp$. Thanks to the second claim, this algorithm is correct because if $x_n \neq \perp$ then $\|x_n - y(b)\| \leq \varepsilon$. Furthermore this algorithm terminates. Indeed, let N be the smallest integer such that

$$L_N \geq 48(k+1)\ell_y(a, b).$$

It exists because $L_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Then $x_N \neq \perp$ and thus the algorithm terminates. Finally, we claim this algorithm has the right complexity. Indeed, let n_0 be the first n such that $x_{n_0} \neq \perp$. By construction, $n_0 \leq N$ and the algorithm computes x_1, x_2, \dots, x_{n_0} and returns. By Lemma 6, the complexity of computing x_n for $n < n_0$ is bounded by

$$\text{poly}\left(k, L_n, \log \|y_0\|, \log \Sigma p, -\log \varepsilon\right)^d$$

since $x_n = \perp$. Furthermore, the complexity of computing x_{n_0} is bounded by

$$\text{poly}\left(k, L_{n_0}, \log \ell_y(a, b), \log \|y_0\|, \log \Sigma p, -\log \varepsilon\right)^d.$$

Since $n_0 \leq N$, it follows that $L_n \leq L_N$ for all $n \leq n_0$ and thus the total complexity is bounded by

$$\sum_{n=1}^{n_0} \text{poly}\left(k, L_N, \log \ell_y(a, b), \log \|y_0\|, \log \Sigma p, -\log \varepsilon\right)^d.$$

Furthermore, since $L_n = n$ and N is the smallest integer such that $L_N \geq 48(k+1)\ell_y(a, b)$, it must be the case that

$$L_N < 49(k+1)\ell_y(a, b)$$

and thus that

$$n_0 \leq N < 49(k+1)\ell_y(a, b).$$

Putting everything together, we get that the total complexity is bounded by

$$\begin{aligned}
& \sum_{n=1}^{n_0} \text{poly} \left(k, 96(k+1)\ell_y(a, b), \log \ell_y(a, b), \log \|y_0\|, \log \Sigma p, -\log \varepsilon \right)^d \\
& \leq \sum_{n=1}^{n_0} \text{poly} \left(k, \ell_y(a, b), \log \|y_0\|, \log \Sigma p, \mu + \ln 3 \right)^d \\
& \leq n_0 \text{poly} \left(k, \ell_y(a, b), \log \|y_0\|, \log \Sigma p, \mu \right)^d \\
& \leq 49(k+1)\ell_y(a, b) \text{poly} \left(k, \ell_y(a, b), \log \|y_0\|, \log \Sigma p, \mu \right)^d \\
& \leq \text{poly} \left(k, \ell_y(a, b), \log \|y_0\|, \log \Sigma p, \mu \right)^d.
\end{aligned}$$

□

Finally, we would like to remind the reader that the existence of a solution y of a PIVP up to a given time is undecidable, see [GBC07] more details. This explains why, in the previous theorem, we have so assume the existence of the solution if we want to have any hope of computing it.

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